A completeness theorem for strong normalization in minimal deduction modulo

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Abstract. Deduction modulo is an extension of first-order predicate logic where axioms are replaced by rewrite rules and where many theories, such as arithmetic, simple type theory and some variants of set theory, can be expressed. An important question in deduction modulo is to find a condition of the theories that have the strong normalization property. Dowek and Werner have given a semantic sufficient condition for a theory to have the strong normalization property: they have proved a "soundness" theorem of the form: if a theory has a model (of a particular form) then it has the strong normalization property. In this paper, we refine their notion of model in a way allowing not only to prove soundness, but also completeness: if a theory has the strong normalization property, then it has a model of this form. The key idea of our model construction is a refinement of Girard's notion of reducibility candidates. By providing a sound and complete semantics for theories having the strong normalization property, this paper contributes to explore the idea that strong normalization is not only a proof-theoretic notion, but also a model-theoretic one.

We say that a theory is consistent if it does not contain a contradiction. Syntacticly, it means that we cannot prove all the formulae of this theory. In order to prove consistency of a theory, we can use semantic tools: it has been shown that having a $\{0, 1\}$-valued model is a sufficient semantic condition for consistency of theories expressed in predicate calculus. Moreover, as proved by Gödel, having a $\{0, 1\}$-valued model is also a necessary condition of consistency for theories expressed in predicate calculus [11].

Deduction modulo [4] is a logical framework, based on Natural Deduction where axioms are replaced by rewrite rules, allowing to express proofs of many theories like arithmetic [8], simple type theory [5], some variants of set theory [6], etc... In this framework, β-reduction of proofs is used to represent their cut elimination, through the Curry-De Bruijn-Howard correspondence. Therefore strong normalization of the β-reduction ensures the cut elimination property of the corresponding theory, and furthermore its consistency. It ensures also the disjunction property, the witness property, soundness of various proof search methods, etc... But strong normalization is a strictly more powerful property than consistency, as there exists consistent theories that don't strongly normalize [14]. Then we may wonder if there is such a sufficient and necessary semantic condition of theories that are strongly normalizing.

The notion of reducibility candidates was first introduced by J.Y. Girard [10], following the work of Tait [17]. We can see *a posteriori*, their work as proofs of strong normalization, obtained by the existence of a $\mathcal{C}$-valued model, where $\mathcal{C}$ is the algebra of reducibility candidates. This work has been extended to, at least, two non-trivial (as they contain strongly normalizing and not strongly normalizing theories) logical frameworks: Pure Type Systems by P.A. Melliès and B. Werner [15], and Deduction modulo by G. Dowek and B. Werner [7].

But what about the converse? We may wonder if having a $\mathcal{C}$-valued model is also a necessary semantic condition for the strong normalization property, *i.e.* if all strongly normalizing theories expressed in Deduction modulo or Pure Type Systems have a $\mathcal{C}$-valued model.

In this paper, we exhibit a new algebra $\mathcal{C}'$ which is a refinement of $\mathcal{C}$ and we prove that having a $\mathcal{C}'$-valued model is still a sufficient semantic condition of strongly normalizing theories expressed in minimal deduction modulo. And moreover, that it is also a necessary semantic condition of strongly normalizing theories.

1 Minimal deduction modulo

1.1 Syntax

We consider a set $\{T_i\}$ of *sorts*, an infinite set of *variables* of each sort, a set $\{f_j\}$ of function symbols, and a set $\{P_k\}$ of predicate symbols, that come with their *rank*. The formation rules for objects and propositions are the usual ones.

- Variables of sort T_i are terms of sort T_i.
- If f_j is a function symbol of rank $\langle T_1, \dots, T_n, U\rangle$ and $t_1, \dots, t_n$ are respectively objects of sort $T_1, \dots, T_n$, then $f_j\ t_1 \dots t_n$ is a term of sort U.
- If P_k is a predicate symbol of rank $\langle T_1, \dots, T_n\rangle$ and $t_1, \dots, t_n$ are respectively objects of sort $T_1, \dots, T_n$, then $P_k\ t_1 \dots t_n$ is an *atomic proposition*.

Propositions are built-up from atomic propositions with the usual connective $\Rightarrow$ and quantifier $\forall$. Remark that, implicitly, quantification in $\forall x.A$ is restricted over the sort of the variable x.

Definition 1 (Terms and Propositions).
We call $\mathcal{O}$ (as objects*), the set of terms:* $t ::= x \mid f\ t \dots t$
We call $\mathcal{P}$, the set of propositions: $A ::= P\ t \dots t \mid A \Rightarrow A \mid \forall x.A$

In this language, proof-terms can contain both term variables (written $x, y, \dots$) and proof variables (written $\alpha, \beta, \dots$). We call $\mathcal{X}$ the set of proof variables and $\mathcal{Y}$ the set of term variables. Notice that $\mathcal{X}$ and $\mathcal{Y}$ have no common element. Terms are written $t, u, \dots$ while proof-terms are written $\pi, \rho, \dots$

Definition 2 (Proof-terms).
We call $\mathcal{T}$, the set of proof-terms: $\pi := \alpha \mid \lambda\alpha.\pi \mid \pi\ \pi' \mid \lambda x.\pi \mid \pi\ t$

Notice that variables α and x are bound in the constructions $\lambda\alpha.\pi$, and $\lambda x.\pi$. α-conversion, free and bound variables are defined as usual.

Each proof-term construction corresponds to a natural deduction rule: terms of the form α express proofs built with the axiom rule, terms of the form $\lambda\alpha.\pi$ and $(\pi\ \pi')$ express proofs built respectively with the introduction and elimination rules of the implication and terms of the form $\lambda x.\pi$ and $(\pi\ t)$ express proofs built with the introduction and elimination rules of the universal quantifier.

We call *neutral* those proof-terms of $\mathcal{T}$ that are not abstractions *i.e.* of the form α, $(\pi\pi')$ or (πt).

1.2 Typing rules

We call *contexts*, lists of declarations $[\alpha : A]$ where α is a proof-variable and A is a proposition, such that each variable in a declaration is different from all the other variables of the context. In this way, we only consider *well formed* contexts, therefore we have to deal with α-conversion, when concatening them: the only proof-variables that two concatened contexts can share have to be declared proofs of equivalent propositions. Notice that as we only declare proof-variables in contexts, the concatenation of two contexts is always a context.

Given a congruence relation on propositions $\equiv$, we define typing rules as usual, in deduction modulo:

$$\frac{}{\Gamma, \alpha : A \vdash_{\equiv} \alpha : B} \quad A \equiv B \quad \text{(axiom)}$$

$$\frac{\Gamma, \alpha : A \vdash_{\equiv} \pi : B}{\Gamma \vdash_{\equiv} \lambda\alpha\ \pi : C} \quad C \equiv A \Rightarrow B \quad (\Rightarrow\text{-intro})$$

$$\frac{\Gamma \vdash_{\equiv} \pi : C \qquad \Gamma \vdash_{\equiv} \pi' : A}{\Gamma \vdash_{\equiv} (\pi\ \pi') : B} \quad C \equiv A \Rightarrow B \quad (\Rightarrow\text{-elim})$$

$$\frac{\Gamma \vdash_{\equiv} \pi : A}{\Gamma \vdash_{\equiv} \lambda x.\pi : B} \quad B \equiv \forall x.A, \quad x \notin FV(\Gamma) \quad (\forall\text{-intro})$$

$$\frac{\Gamma \vdash_{\equiv} \pi : B}{\Gamma \vdash_{\equiv} \pi\ t : C} \quad B \equiv \forall x.A, \quad C \equiv (t/x)A, \quad t \text{ has the sort of } x \ (\forall\text{-elim})$$

Fig. 1. Typing rules

1.3 Proof reduction rules and strong normalization

In deduction modulo, the process of cut elimination is modeled by β-reduction. We consider the contextual closure of the reduction rules given figure 2. These rules correspond to proof reduction in natural deduction.

$$(\lambda\alpha.\pi\ \pi') \to (\pi'/\alpha)\pi$$
$$(\lambda x.\pi\ t) \to (t/x)\pi$$

Fig. 2. Proof reduction rules

We write $(\pi'/\alpha)\pi$ (resp. $(t/x)\pi$) the substitution of α (resp. x) by π' (resp. t) in π. We write $\pi \to \pi'$ if π reduces in one step to π', $\pi \to^+ \pi'$ if π reduces in at least one step to π', and $\pi \to^* \pi'$ if π reduces in an arbitrary number of steps to π'.
A proof-term is said to be *normal* if it contains no redex and *strongly normalizing* if all reduction sequences issued from this proofs are finite. We write SN for the set of strongly normalizing proofs.

Definition 3 (Isolated proof-terms). *A proof-term is called* isolated *if it is neutral and only reduces to neutral terms (*i.e *it never reduces to an abstraction, in any number of reduction steps).*

1.4 Theories expressed in minimal deduction modulo

A theory expressed in minimal deduction modulo is defined by a many-sorted language in predicate logic $\mathcal{L} = \langle \{T_i\}, \{f_j\}, \{P_k\} \rangle$ and a congruence relation $\equiv$ on propositions of the associated many-sorted logic.

Remark 1. Given a theory $\mathcal{L}_\equiv$, we will write $\vdash$ for $\vdash_\equiv$.

Proposition 1 (confluence and subject-reduction). $\to$ *is confluent.*
And for all contexts Γ, proof-terms π, π' and propositions A,
if $\Gamma \vdash \pi : A$ and $\pi \to \pi'$, then $\Gamma \vdash \pi' : A$.

Example As mentioned above, deduction modulo allows to express (intentional) simple type theory [1] without any axiom.

We show in the following, how minimal deduction modulo permits to express minimal (intentional) simple type theory, without any axiom (see [5] for details).

The *sorts* are *simple types* inductively defined by:
- ι and o are sorts,
- if T and U are sorts then $T \to U$ is a sort.

The language is composed of the individual symbols
- $S_{T,U,V}$ of sort $(T \to U \to V) \to (T \to U) \to T \to V$,
- $K_{T,U}$ of sort $T \to U \to T$,
- $\dot{\Rightarrow}$, of sort o,
- $\dot{\forall}_T$ of sort $(T \to o) \to o$,

the function symbols $\alpha_{T,U}$ of rank $\langle T \to U, T, U\rangle$,
and the predicate symbol ε of rank $\langle o\rangle$.

The combinators $S_{T,U,V}$ and $K_{T,U}$ are used to express functions. The objects $\dot{\Rightarrow}$, and $\dot{\forall}_T$ allow to represent propositions as objects of sort o. Finally, the predicate ε allows to transform such an object t of type o into the actual corresponding proposition $\varepsilon(t)$.

$$\begin{aligned}
\alpha(\alpha(\alpha(S_{T,U,V}, x), y), z) &\to \alpha(\alpha(x, z), \alpha(y, z)) \\
\alpha(\alpha(K_{T,U}, x), y) &\to x \\
\varepsilon(\alpha(\alpha(\dot{\Rightarrow}, x), y)) &\to \varepsilon(x) \Rightarrow \varepsilon(y) \\
\varepsilon(\alpha(\dot{\forall}, x)) &\to \forall y\ \varepsilon(\alpha(x, y))
\end{aligned}$$

2 Language-dependent truth values algebras

Truth values algebras (TVAs) are an extension of Heyting algebras, defined by G. Dowek in [3], which provide an algebraic universe to study consistency and cut elimination of theories expressed in deduction modulo. We introduce in this paper, a new category of algebras : *language-dependent truth values algebras* (LDTVAs) which are both a simplification and a refinement of TVAs.

2.1 Definition

For all sets E, we call $\mathbb{P}(E)$ the set of subsets of E.
For all sorts T of a language $\mathcal{L}$, we write $\hat{T}$, the set of closed terms of sort T.

Definition 4 (language-dependent tvas).
Let $\mathcal{L} = \langle\{T_i\}, \{f_j\}, \{P_k\}\rangle$ be a many-sorted language in predicate logic.
$\langle\mathcal{B}, \dot{\Rightarrow}, (\hat{\mathcal{A}}_T), (\hat{\forall}_T)\rangle$ is a LDTVA *for $\mathcal{L}$ if and only if:*

- *$\mathcal{B}$ is a set (called the* domain*),*
- *$\dot{\Rightarrow}$ is a function from $\mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$,*
- *for all sorts T,*
  - *$\hat{\mathcal{A}}_T$ is a set of functions from $\hat{T}$ to $\mathcal{B}$: $\hat{\mathcal{A}}_T \subseteq \hat{T} \mapsto \mathcal{B}$*
  - *$\hat{\forall}_T$ is a function from $\hat{\mathcal{A}}_T$ to $\mathcal{B}$.*

Definition 5 (Valuation).
Given a LDTVA *for $\mathcal{L} = \langle\{T_i\}, \{f_j\}, \{P_k\}\rangle$, a* valuation *$\varphi$ is a substitution mapping term-variables of a sort T_i to closed terms of sort T_i. For all propositions A (resp. terms t), we call* $\text{VAL}(A)$ *(resp.* $\text{VAL}(t)$*) the set of valuations whose domain contains the set of free variables of A (resp. t).*
We write $\text{DOM}(\varphi)$*, the domain of a valuation φ.*

Definition 6. *For all $A \in \mathcal{P}$, terms t and $\varphi \in \text{VAL}(A)$, we write A_φ the result of the substitution φ on A.*

Definition 7 (Interpretations). *We call* $\mathcal{B}$-valued interpretations *those functions which map every ordered pair of a proposition A and a valuation in* $\mathrm{VAL}(A)$ *to an element of the domain of a* LDTVA $\mathcal{B}$.

Definition 8 (Models).
Let $\mathcal{L} = \langle \{T_i\}, \{f_j\}, \{P_k\} \rangle$ *be a many-sorted language in predicate logic,*
let $\equiv$ *be a congruence relation on propositions of minimal deduction based on* $\mathcal{L}$,
let $\mathcal{B} = \langle \mathcal{B}, \hat{\Rightarrow}, (\hat{\mathcal{A}}_T), (\hat{\forall}_T) \rangle$ *be a* LDTVA *for* $\mathcal{L}$.

1. *A $\mathcal{B}$-valued interpretation* $[\![.]\!]_.$ *is a* $\mathcal{B}$-valued model *if and only if:*
   - *for all* $A, B \in \mathcal{P}$ *and* $\varphi \in \mathrm{VAL}(A \Rightarrow B)$, $[\![A \Rightarrow B]\!]_\varphi = [\![A]\!]_\varphi \mathrel{\hat{\Rightarrow}} [\![B]\!]_\varphi$
   - *for all* $A \in \mathcal{P}$, x *of sort* T *and* $\varphi \in \mathrm{VAL}(\forall x.A)$ *such that* $x \notin \mathrm{DOM}(\varphi)$, $[\![\forall x.A]\!]_\varphi = \hat{\forall}_T(t \mapsto [\![A]\!]_{\varphi + \langle x,t \rangle})$
   - *for all* $A \in \mathcal{P}$, x *of sort* T, $t \in \hat{T}$ *and* $\varphi \in \mathrm{VAL}(\forall x.A)$ *such that* $x \notin \mathrm{DOM}(\varphi)$, $[\![(t/x)A]\!]_\varphi = [\![A]\!]_{\varphi + \langle x,t \rangle}$.
2. *A $\mathcal{B}$-valued model* $[\![.]\!]_.$ *is a* model of the theory $\mathcal{L}_\equiv$ *if and only if:*
   for all $A, A' \in \mathcal{P}$, $\varphi \in \mathrm{VAL}(A)$ *and* $\psi \in \mathrm{VAL}(A')$, *if* $A_\varphi \equiv A'_\psi$, *then* $[\![A]\!]_\varphi = [\![A']\!]_\psi$

Remark 2. The previous conditions can be reformulated as: 1. Interpretations of propositions have to be adapted to the connectives to be a model. 2. Models have to be adapted to the congruence to be a model of the associated theory.

LDTVAs are first a simplification, of TVAs because of the fact that we consider theories with rewrite rules only and no axioms, therefore we don't have to care about the so-called *positive* truth values [3]. The refinement comes from the interpretation of the universal quantifier: $\hat{\forall}$.

In TVAs, $\hat{\forall}$ is defined as a function mapping subsets of the domain of the algebra $\mathcal{B}$, to $\mathcal{B}$. Given a substitution φ that maps term-variables to $\hat{T}$ (the domain of T), the interpretation $[\![\forall x.A]\!]_\varphi$ of a proposition $\forall x.A$, is obtained by applying $\hat{\forall}$ to the set $\{[\![A]\!]_{\varphi+\langle x,v \rangle},\ v \in \hat{T}\}$. The function $\hat{\forall}$ is typically a greatest lower bound or an intersection.That is sufficient to build a notion of model valued on $\mathcal{C}$, such that having such a model is a sufficient condition for a theory to be strongly normalizing. But it creates difficulties when we want to prove that this condition is necessary.

In LDTVAs, we use a more usual ([9]) operator $\hat{\forall}$: this operator is a function mapping functions from $\hat{T}$ to $\mathcal{B}$, to elements of $\mathcal{B}$. In this case $[\![\forall x.A]\!]_\varphi$ is obtained by applying $\hat{\forall}$ to the function that maps elements v of $\hat{T}$ to $[\![A]\!]_{\varphi+\langle x,v \rangle}$.

In the case of the reducibility candidates algebra $\mathcal{C}$, the domain consists of sets of proof-terms and $[\![\forall x.A]\!]_\varphi$ is defined as the set of proof-terms π such that for all $t \in \hat{T}$, πt is in $\bigcap\{[\![A]\!]_{\varphi+\langle x,t' \rangle},\ t' \in \hat{T}\}$. The main properties that ensure the fact that having a $\mathcal{C}$-valued model is a suffcient condition for a theory to be strongly normalizing, are that all elements of the domain contains only strongly normalizing proof-terms, and that all proofs of a proposition are in their interpretation. In order to make this condition be also necessary, a solution is to make interpretations of propositions contain only their proofs.

In the particular case where $\hat{T}$ is the set of closed terms of sort T, the definition of $\hat{\forall}$ leads to interpret too strictly propositions of the form $\forall x.A$. For example, if P is an unary atomic proposition, then $\pi = \lambda x.\lambda\alpha.(\alpha\ x)$ is a proof of $A = \forall x.\ (\forall y.\ P(y)) \Rightarrow P(x)$, but π is not in the interpretation of A as for all $t \neq t'$, πt is not a proof of $(\forall y.\ P(y)) \Rightarrow P(t')$.

In LDTVAS, we choose to always take for each $\hat{T}$ the set of closed terms of sort T. Therefore LDTVAS depend on the language they are built on. And we will see, in the following, how our new definition allows us to define $[\![\forall x.A]\!]_\varphi$ as a "dependent intersection", *i.e.* the set of proof-terms π such that for all $t \in \hat{T}$, πt is in $[\![A]\!]_{\varphi+\langle x,t\rangle}$, and therefore capture exactly proofs of $\forall x.A_\varphi$.

Definition 9 (Morphism).
Let $\mathcal{B}^1 = \langle \mathcal{B}^1, \Rightarrow^1, (\hat{\mathcal{A}}^1_T), (\hat{\forall}^1_T)\rangle\rangle$ and $\mathcal{B}^2 = \langle \mathcal{B}^2, \Rightarrow^2, (\hat{\mathcal{A}}^2_T), (\hat{\forall}^2_T)\rangle$ be two LDTVA*s. A* morphism *from $\mathcal{B}^1$ to $\mathcal{B}^2$ is a function F from $\mathcal{B}^1$ to $\mathcal{B}^2$ such that:*

- *for all $E, G \in \mathcal{B}^1$, $F(E \Rightarrow^1 G) = F(E) \Rightarrow^2 F(G)$,*
- *for all sorts T, $x \in \hat{T}$ and $f \in \mathring{\mathcal{A}}_T$, $F(\hat{\forall}^1_T\ f) = \hat{\forall}^2_T\ F \circ f$.*

Lemma 1. *For all* LDTVA*s $\mathcal{B}_1$ and $\mathcal{B}_2$ and morphisms F from $\mathcal{B}_1$ to $\mathcal{B}_2$, if $[\![.]\!]_.$ is a $\mathcal{B}_1$-valued model of a theory $\mathcal{L}_\equiv$, then $F \circ [\![.]\!]_.$ is a $\mathcal{B}_2$-valued model of $\mathcal{L}_\equiv$.*

3 About reducibility candidates and typing

In order to build a sufficient and necessary semantic condition for strongly normalizing theories expressed in minimal deduction modulo, we use the following method: we first define, for each theory $\mathcal{L}_\equiv$ a LDTVA $\mathcal{C}_\equiv$, wich depends on $\equiv$ such that having a $\mathcal{C}_\equiv$-valued model is a necessary condition to be strongly normalizing. Then we define another LDTVA $\mathcal{C}'$, wich does not depend on $\equiv$ anymore and a morphism from each $\mathcal{C}_\equiv$ to $\mathcal{C}'$, such that having a $\mathcal{C}'$-valued model is also a necessary condition to be strongly normalizing. Finally, we prove that this condition is also sufficient.

For the following, we set a theory $\mathcal{L}_\equiv$.

3.1 $\mathcal{C}_\equiv$, a ldtva of ($\equiv$) well-typed reducibility candidates

As a typing judgement $\Gamma \vdash \pi : A$ associates an ordered pair formed by a context Γ and a proof π , to a proposition A, we will consider, for the domain of $\mathcal{C}_\equiv$, the set of sets of such pairs, and interpret propositions by the pairs they are associated to. We write $\mathcal{U}$ the set of such pairs. We will consider, for the domain of $\mathcal{C}_\equiv$ (wich we will also call $\mathcal{C}_\equiv$), the set of subsets of $\mathcal{U}$ which verify adapted versions of the usual properties (CR_1) and (CR_2) of reducibility candidates, a modified version of (CR_3) and another property ($\text{CR}_\equiv$), which express both well-typing.

In usual reducibility candidates, the property called (CR_3) expresses the fact that if a proof-term π is neutral and all its one-step reducts are in a candidate C, then π is also in C. As a normal proof-term has no one-step reducts, all normal neutral proof-terms belong to all reducibility candidates. In particular, for all proof-variables α, $\alpha\alpha$ belong to all reducibility candidates while it is not well typed in any theory which is strongly normalizing. In order to avoid proof-terms that are not well-typed, (*i.e.* proof-terms that are not proofs of the interpreted proposition), we propose a modified version of (CR_3): ($CR_{3_\equiv}$), wich excludes explicitely proof-terms that are not well-typed.

Definition 10 **($\mathcal{U}$).**
$\mathcal{U} = \{(\Gamma, \pi)$ such that Γ is a context and π is a proof-term $\}$.

Definition 11.
For all $E \subseteq \mathcal{U}$, we define the following properties :

- *($CR_\equiv$) There exists A_E such that $\forall(\Gamma, \pi) \in E,\ \Gamma \vdash \pi : A_E$*
- *($CR_{1_\equiv}$) For all $(\Gamma, \pi) \in E$, $\pi \in SN$*
- *($CR_{2_\equiv}$) For all $(\Gamma, \pi) \in E$, and $\pi' \in \mathcal{T}$ such that $\pi \to \pi'$, $(\Gamma, \pi') \in E$*
- *($CR_{3_\equiv}$) For all $(\Gamma, \pi) \in \mathcal{U}$ such that π is neutral and $\Gamma \vdash \pi : A_E$, if for all one-step reducts τ of π, $(\Gamma, \tau) \in E$, then $(\Gamma, \pi) \in E$.*

Remark 3. For all $E \subseteq \mathcal{U}$, if E satisfies ($CR_\equiv$) and ($CR_{3_\equiv}$), then for all proof-variables α, $(\alpha : A_E, \alpha) \in E$, but $(\Gamma, \alpha\alpha) \notin E$, for any context Γ, if $\mathcal{L}_\equiv$ is strongly normalizing.

Definition 12 **(domain $\mathcal{C}_\equiv$).** *We call $\mathcal{C}_\equiv$ the set of subsets of $\mathcal{U}$ which satisfy ($CR_\equiv$), ($CR_{1_\equiv}$), ($CR_{2_\equiv}$) and ($CR_{3_\equiv}$).*

Then we adapt the usual interpretation of $\Rightarrow$ to elements of $\mathcal{U}$.

Definition 13 **($\mathring{\Rightarrow}$).** *For all $E, F \subseteq \mathcal{U}$,*
$E \mathring{\Rightarrow} F = \{(\Gamma, \pi) \in \mathcal{U}$ such that for all $(\Gamma', \pi') \in E$, $(\Gamma\Gamma', \pi\pi') \in F\}$

Remark 4. We recall the fact that we only consider well-formed contexts, therefore the only variables Γ and Γ' can share have to be declared proofs of equivalent propositions, otherwise we have to deal with α-conversion when concatening Γ and Γ' into $\Gamma\Gamma'$.

Lemma 2. *$\mathring{\Rightarrow}$ is a function from $\mathcal{C}_\equiv \times \mathcal{C}_\equiv$ to $\mathcal{C}_\equiv$.*

Proof. Let $E, F \in \mathcal{C}_\equiv$, and $(\Gamma, \pi) \in E \mathring{\Rightarrow} F$,

- ($CR_\equiv$) Let α be a proof-variable. As E satisfies ($CR_\equiv$) and ($CR_{3_\equiv}$), $(\alpha : A_E, \alpha) \in E$, therefore $(\Gamma, \alpha : A_E, \pi\alpha) \in F$, as $(\Gamma, \pi) \in E \mathring{\Rightarrow} F$. As F satisfies ($CR_\equiv$), we have $\Gamma, \alpha : A_E \vdash \pi\alpha : A_F$. Therefore $\Gamma \vdash \pi : A_E \Rightarrow A_F$ (by case on the last rule used in the derivation of $\Gamma, \alpha : A_E \vdash \pi\alpha : A_F$). Finally, $A_{E \mathring{\Rightarrow} F} \equiv A_E \Rightarrow A_F$.
- ($CR_{1_\equiv}$) Let α be a proof-variable. As E satisfies ($CR_\equiv$) and ($CR_{3_\equiv}$), $(\alpha : A_E, \alpha) \in E$, therefore $(\Gamma, \alpha : A_E, \pi\alpha) \in F$, as $(\Gamma, \pi) \in E \mathring{\Rightarrow} F$. As F satisfies ($CR_{1_\equiv}$), we have $\pi\alpha \in SN$, therefore $\pi \in SN$.

- ($CR_{2_\equiv}$) Let τ such that $\pi \to \tau$. Then, for all $(\Gamma', \pi') \in E$, $(\Gamma\Gamma', \pi\pi') \in F$ and $\pi\pi' \to \tau\pi'$, therefore $(\Gamma\Gamma', \tau\pi') \in F$ as F satisfies ($CR_{2_\equiv}$). Finally, $(\Gamma, \tau) \in E \mathring{\Rightarrow} F$.
- ($CR_{3_\equiv}$) Let $(\Gamma, \mu) \in \mathcal{U}$ such that $\Gamma \vdash \mu : A_{E \mathring{\Rightarrow} F}$, μ is neutral and all its one-step reducts are in $E \mathring{\Rightarrow} F$. Then, $\Gamma \vdash \mu : A_E \Rightarrow A_F$. For all $(\Gamma', \pi') \in E$, $\mu\pi'$ is neutral and we have $\Gamma\Gamma' \vdash \mu\pi' : A_F$. Let τ be a one-step reduct of $\mu\pi'$. We prove, by induction on the maximal length of a reduction sequence from π' ($\in SN$), that $(\Gamma\Gamma', \tau) \in F$. As μ is neutral, either $\tau = \mu\pi''$ with $\pi' \to \pi''$, and we conclude by induction hypothesis. Either $\tau = \mu'\pi'$, and in this case, $(\Gamma\Gamma', \tau) \in F$, by hypothesis on μ. Finally, for all $(\Gamma', \pi') \in E$, $(\Gamma\Gamma', \mu\pi') \in F$, as F satisfies ($CR_{3_\equiv}$), and finally $(\Gamma, \mu) \in E \mathring{\Rightarrow} F$.

Definition 14 **($\mathring{\mathcal{A}}_T$).** *For all sorts T,*
$\mathring{\mathcal{A}}_T = \{f : \hat{T} \mapsto \mathcal{C}_\equiv$*, such that there exists $A_f \in \mathcal{P}$ and $x_f \in \mathcal{X}$ such that for all $t \in \hat{T}$ and $(\Gamma, \pi) \in f(t)$,* $\Gamma \vdash \pi : (t/x_f)A_f\}$

Remark 5. In other words, $f \in \mathring{\mathcal{A}}_T$ iff for all $t \in \hat{T}$, $A_{f(t)} = (t/x_f)A_f$.

Now we can define what we called "dependent intersection" previously.

Definition 15 **($\mathring{\forall}_T$).** *For all sorts T and functions $f \in \mathring{\mathcal{A}}_T$,*
$\mathring{\forall}_T f = \{(\Gamma, \pi) \in \mathcal{U}$ *such that for all $t \in \hat{T}$, $(\Gamma, \pi t) \in f(t)\}$*

Lemma 3. *For all sorts T, $\mathring{\forall}_T$ is a function from $\mathring{\mathcal{A}}_T$ to $\mathcal{C}_\equiv$.*

Proof. Let $f \in \mathring{\mathcal{A}}_T$, and $(\Gamma, \pi) \in \mathring{\forall}_T f$

- ($CR_\equiv$) Let $t \in \hat{T}$ ($\neq \emptyset$). Then $(\Gamma, \pi t) \in f(t)$. As $f \in \mathring{\forall}_T$, we have $\Gamma \vdash \pi t : (t/x_f)A_f$. Therefore $\Gamma \vdash \pi : \forall x_f.A_f$, by case on the last rule used in the derivation of $\Gamma \vdash \pi t : (t/x_f)A_f$. Finally, $A_{\mathring{\forall}_T f} \equiv \forall x_f.A_f$.
- ($CR_{1_\equiv}$) Let $t \in \hat{T}$ ($\neq \emptyset$). Then $(\Gamma, \pi t) \in f(t) \in \mathcal{C}_\equiv$ therefore $\pi t \in SN$ and so does π.
- ($CR_{2_\equiv}$) Let π' such that $\pi \to \pi'$. Then, for all $t \in \hat{T}$, $\pi t \to \pi' t$, therefore $\pi' t \in f(t) \in \mathcal{C}_\equiv$.
- ($CR_{3_\equiv}$) Let $(\Gamma, \mu) \in \mathcal{U}$ such that μ is neutral, $\Gamma \vdash \mu : \forall x_f.A_f$, and for all one-step reducts μ' of μ, $(\Gamma, \mu') \in \mathring{\forall}_T f$. Let $t \in \hat{T}$, then μt is neutral, $\Gamma \vdash \mu t : (t/x_f)A_f$, and, as μ is neutral, all one-step reducts of μt are of the form $\mu' t$, with $\mu \to \mu'$, hence $(\Gamma, \mu t) \in f(t)$ as $f(t)$ satisfies ($CR_{3_\equiv}$). Finally, $(\Gamma, \mu) \in \mathring{\forall}_T f$.

Definition 16 **($\mathcal{C}_\equiv$).** *$\mathcal{C}_\equiv$ is the* LDTVA $\langle \mathcal{C}_\equiv, \mathring{\Rightarrow}, (\mathring{\mathcal{A}}_T), (\mathring{\forall}_T) \rangle$.

3.2 Building a $\mathcal{C}_\equiv$-valued interpretation

Let us now define a first $\mathcal{C}_\equiv$-valued model, by using directly definitions of $\mathring{\Rightarrow}$ and $\mathring{\forall}_T$, and well-chosen interpretations of atomic propositions A (proofs of A wich are strongly normalizing).

Definition 17. *Let A be a proposition and $\varphi \in \mathrm{VAL}(A)$.*
We define the subset of $\mathcal{U}$, $[A]_\varphi$ by induction over the structure of A.

- $[P\ t_1 \ldots t_n]_\varphi = \{(\Gamma, \pi) \in \mathcal{U}$ *such that* $\pi \in SN$ *and* $\Gamma \vdash \pi : (P\ t_1 \ldots t_n)_\varphi\}$
- $[B \Rightarrow C]_\varphi = [B]_\varphi \mathring{\Rightarrow} [C]_\varphi$
- $[\forall x.B]_\varphi = \mathring{\forall}_T\ (t \mapsto [B]_{\varphi+\langle x,t\rangle})$

Lemma 4. *For all $A \in \mathcal{P}$, x of sort T, $t \in \hat{T}$ and $\varphi \in \mathrm{VAL}(\forall x.A)$ such that $x \notin \mathrm{DOM}(\varphi)$, we have $[(t/x)A]_\varphi = [A]_{\varphi+\langle x,t\rangle}$.*

Proof. By induction on A.

Lemma 5. *For all $A \in \mathcal{P}$, and $\varphi \in \mathrm{VAL}(A)$,*

$[A]_\varphi \in \mathcal{C}_\equiv$ *with* $A_{[A]_\varphi} = A_\varphi$ *(i.e, for all $(\Gamma, \pi) \in [A]_\varphi$, $\Gamma \vdash \pi : A_\varphi$).*

Proof. By induction on A.

- If A is an atomic proposition $P\ t_1 \ldots t_n$,
  ($\mathrm{CR}_\equiv$) By definition. (with $A_{[P\ t_1 \ldots t_n]_\varphi} \equiv P\ \varphi(t_1) \ldots \varphi(t_n)$).
  ($\mathrm{CR1}_\equiv$) By definition.
  ($\mathrm{CR2}_\equiv$) By subject-reduction and the fact that each reduct of a proof-term in SN is also in SN.
  ($\mathrm{CR3}_\equiv$) If all one-step reducts of a proof-term are in SN, then it is also in SN.
- If $A = B \Rightarrow C$, as $\mathring{\Rightarrow} : \mathcal{C}_\equiv \times \mathcal{C}_\equiv \mapsto \mathcal{C}_\equiv$, we conclude by induction hypothesis (with $A_{[B \Rightarrow C]_\varphi} = A_{[B]_\varphi \mathring{\Rightarrow} [C]_\varphi} \equiv A_{[B]_\varphi} \Rightarrow A_{[C]_\varphi} \equiv B_\varphi \Rightarrow C_\varphi = (B \Rightarrow C)_\varphi$).
- If $A = \forall x.B$, let T be the sort of x and $f = t \mapsto [B]_{\varphi+\langle x,t\rangle}$.
  Then f is a function from $\hat{T}$ to $\mathcal{C}_\equiv$, by induction hypothesis. Moreover, for all $t \in \hat{T}$, $A_{f(t)} = B_{\varphi+\langle x,t\rangle} = (t/x)B_\varphi$, by induction hypothesis. Therefore $f \in \mathring{\mathcal{A}}_T$ and $\mathring{\forall}_T f \in \mathcal{C}_\equiv$ (with $A_{[\forall x.B]_\varphi} = \forall x.A_f = (\forall x.B)_\varphi$).

At this point, we have $\mathcal{C}_\equiv$-valued model which is adapted to typing: the interpretation of a proposition only contains proofs of this proposition. But it is not necessarily adapted to the congruence relation. Indeed, in a theory where we have two atomic proposition symbols P and Q such that $P \equiv (Q \Rightarrow Q)$ (notice that such a theory can be strongly normalizing), then for all valuations $\varphi \in \mathrm{VAL}(P) \cap \mathrm{VAL}(Q)$, $[P]_\varphi \neq [Q]_\varphi \mathring{\Rightarrow} [Q]_\varphi$. We have then to modify this interpretation to make it a $\mathcal{C}_\equiv$-valued model of $\mathcal{L}_\equiv$.

3.3 Adapting this interpretation to the congruence

We simply force the adaptation to the congruence, in the following definition:

Definition 18. *We define a second interpretation $\lfloor . \rfloor_.$, as follows :*
for all $A \in \mathcal{P}$ and $\varphi \in \mathrm{VAL}(A)$,

$$\lfloor A \rfloor_\varphi = \bigcap_{A_\varphi \equiv A'_\psi} [A']_\psi$$

Remark 6. For all $A, A' \in \mathcal{P}$, $\varphi \in \mathrm{VAL}(A)$ and $\psi \in \mathrm{VAL}(A')$ such that $A_\varphi \equiv A'_\psi$, we have $\lfloor A \rfloor_\varphi = \lfloor A' \rfloor_\psi$, by definition.

Then we prove that $\lfloor . \rfloor_.$ is also a $\mathcal{C}_\equiv$-valued interpretation adapted to typing.

Lemma 6. *For all $A \in \mathcal{P}$, and $\varphi \in \mathrm{VAL}(A)$,*
$\lfloor A \rfloor_\varphi \in \mathcal{C}_\equiv$ *with* $A_{\lfloor A \rfloor_\varphi} = A_\varphi$ *(i.e, $\forall (\Gamma, \pi) \in \lfloor A \rfloor_\varphi$, $\Gamma \vdash \pi : A_\varphi$).*

Proof. By lemma 5.

Lemma 7. *For all $A \in \mathcal{P}$, x of sort T, $t \in \hat{T}$ and $\varphi \in \mathrm{VAL}(\forall x.A)$ such that $x \notin \mathrm{DOM}(\varphi)$, we have $\lfloor (t/x)A \rfloor_\varphi = \lfloor A \rfloor_{\varphi + \langle x, t \rangle}$.*

Proof. By lemma 4.

Finally, we proved, that $\lfloor . \rfloor_.$ is a $\mathcal{C}_\equiv$-valued interpretation of propositions adapted to typing and to the congruence relation $\equiv$. Let us now show that if the theory $\mathcal{L}_\equiv$ is strongly normalizing, then $\lfloor . \rfloor_.$ is a $\mathcal{C}_\equiv$-valued model of $\mathcal{L}_\equiv$, *i.e.* it is also adapted to connectives.

3.4 $\lfloor . \rfloor_.$ is a $\mathcal{C}_\equiv$-valued model of strongly normalizing theories $\mathcal{L}_\equiv$

In order to prove that $\lfloor . \rfloor_.$ is a $\mathcal{C}_\equiv$-valued model of $\mathcal{L}_\equiv$, if it is strongly normalizing, we proceed by contraposition, showing that if $\lfloor . \rfloor_.$ is not connectives-adapted, then we can exhibit a typing judgement $\Gamma \vdash \pi : A$ such that $\pi \notin SN$.

Lemma 8.
If there exists $A, B \in \mathcal{P}$ and $\varphi \in \mathrm{VAL}(A \Rightarrow B)$, such that $\lfloor A \Rightarrow B \rfloor_\varphi \neq \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi$ then there exists $\pi \in \mathcal{T}$, $C \in \mathcal{P}$, $\psi \in \mathrm{VAL}(C)$ such that $\Gamma \vdash \pi : C_\psi$ and $(\Gamma, \pi) \notin \lfloor C \rfloor_\psi$.

Proof.
- If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \notin \lfloor A \Rightarrow B \rfloor_\varphi$ and $(\Gamma, \pi) \in \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi$. Then $\Gamma \vdash \pi : A_\varphi \Rightarrow B_\varphi = (A \Rightarrow B)_\varphi$.
  We take $C = A \Rightarrow B$ and $\psi = \varphi$.
- If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \in \lfloor A \Rightarrow B \rfloor_\varphi$ and $(\Gamma, \pi) \notin \lfloor A \rfloor_\varphi \mathring{\Rightarrow} \lfloor B \rfloor_\varphi$. Then there exists $(\Gamma', \pi') \in \lfloor A \rfloor_\varphi$ such that $(\Gamma\Gamma', \pi\pi') \notin \lfloor B \rfloor_\varphi$. As $(\Gamma, \pi) \in \lfloor A \Rightarrow B \rfloor_\varphi$, and $(\Gamma', \pi') \in \lfloor A \rfloor_\varphi$, we have $\Gamma \vdash \pi : (A \Rightarrow B)_\varphi = A_\varphi \Rightarrow B_\varphi$ and $\Gamma' \vdash \pi' : A_\varphi$. Therefore $\Gamma\Gamma' \vdash \pi\pi' : B_\varphi$.
  We take $C = A \Rightarrow B$ and $\psi = \varphi$.

Lemma 9.
If there exists $A \in \mathcal{P}$, $\varphi \in \mathrm{VAL}(A)$, and x of sort T such that $x \notin \mathrm{DOM}(\varphi)$, and
$$\lfloor \forall x.A \rfloor_\varphi \neq \mathring{\forall}_T (t \mapsto \lfloor A \rfloor_{\varphi + \langle x, t \rangle})$$
then there exists $\pi \in \mathcal{T}$, $C \in \mathcal{P}$, $\psi \in \mathrm{VAL}(C)$ such that $\Gamma \vdash \pi : C_\psi$ and $(\Gamma, \pi) \notin \lfloor C \rfloor_\psi$.

Proof.
- If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \notin \lfloor \forall x.A \rfloor_\varphi$ and $(\Gamma, \pi) \in \mathring{\forall}_T (t \mapsto \lfloor A \rfloor_{\varphi + \langle x, t \rangle})$. Then $\Gamma \vdash \pi : \forall x.A_\varphi$.
  We take $C = \forall x.A$ and $\psi = \varphi$.

– If there exists $(\Gamma,\pi)\in\mathcal{U}$ such that $(\Gamma,\pi)\in\lfloor\forall x.A\rfloor_\varphi$ and $(\Gamma,\pi)\notin\mathring{\forall}_T(t\mapsto\lfloor A\rfloor_{\varphi+\langle x,t\rangle})$. Then there exists $t\in\hat{T}$ such that $(\Gamma,\pi t)\notin\lfloor A\rfloor_{\varphi+\langle x,t\rangle})$. As $(\Gamma,\pi)\in\lfloor\forall x.A\rfloor_\varphi$, we have $\Gamma\vdash\pi:\forall x.A_\varphi$, therefore $\Gamma\vdash\pi t:(t/x)A_\varphi=A_{\varphi+\langle x,t\rangle}$. We take $C=A$ and $\psi=\varphi+\langle x,t\rangle$

Lemma 10.
If there exists $A,B\in\mathcal{P}$, $\varphi\in\mathrm{VAL}(A\Rightarrow B)$ or $\varphi'\in\mathrm{VAL}(\forall x.A)$ with x of sort T, $x\notin\mathrm{DOM}(\varphi')$ and

$$\lfloor A\Rightarrow B\rfloor_\varphi\neq\lfloor A\rfloor_\varphi\mathring{\Rightarrow}\lfloor B\rfloor_\varphi\quad\text{or}\quad\lfloor\forall x.A\rfloor_{\varphi'}\neq\mathring{\forall}_T(t\mapsto\lfloor A\rfloor_{\varphi'+\langle x,t\rangle})$$

then there exists $D\in\mathcal{P}$, $\pi\in\mathcal{T}$, $\psi\in\mathrm{VAL}(D)$ such that $\Gamma\vdash\pi:D_\psi$ and $(\Gamma,\pi)\notin[D]_\psi$.

Proof. By lemmas 8 and 9, there exists C, Γ, π and ψ such that $\Gamma\vdash\pi:C_\psi$ and $(\Gamma,\pi)\notin\lfloor C\rfloor_\psi$. Therefore, there exists a proposition D and $\psi'\in\mathrm{VAL}(D)$ such that $D_{\psi'}\equiv C_\psi$ and $(\Gamma,\pi)\notin[D]_{\psi'}$. And $\Gamma\vdash\pi:D_{\psi'}$, by equivalence of C_ψ and $D_{\psi'}$.

Lemma 11.
If there exists $A,B\in\mathcal{P}$, $\varphi\in\mathrm{VAL}(A\Rightarrow B)$ or $\varphi'\in\mathrm{VAL}(\forall x.A)$ with x of sort T, $x\notin\mathrm{DOM}(\varphi')$

$$\text{and}\qquad\lfloor A\Rightarrow B\rfloor_\varphi\neq\lfloor A\rfloor_\varphi\mathring{\Rightarrow}\lfloor B\rfloor_\varphi\quad\text{or}\quad\lfloor\forall x.A\rfloor_{\varphi'}\neq\mathring{\forall}_T(t\mapsto\lfloor A\rfloor_{\varphi'+\langle x,t\rangle})$$

then there exists a (term-closed) proposition E, $\pi\in\mathcal{T}$ and a context Γ such that

$$\Gamma\vdash\pi:E\ \text{and}\ \pi\notin SN.$$

Proof. By lemma 10, there exists a proposition D, a context Γ, a proof π and $\varphi\in\mathcal{V}(D)$ such that $\Gamma\vdash\pi:D_\varphi$ and $(\Gamma,\pi)\notin[D]_\varphi$. By induction on D.

– if D is atomic, then as $\Gamma\vdash\pi:D_\varphi$, we have $\pi\notin SN$.
– if $D=F\Rightarrow G$,
  then $\Gamma\vdash\pi:(F\Rightarrow G)_\varphi$ and $(\Gamma,\pi)\notin[F\Rightarrow G]_\varphi=[F]_\varphi\mathring{\Rightarrow}[G]_\varphi$. Then there exists $(\Gamma',\pi')\in[F]_\varphi$ such that $(\Gamma\Gamma',\pi\pi')\notin[G]_\varphi$. As $(\Gamma,\pi)\in[F\Rightarrow G]_\varphi$, and $(\Gamma',\pi')\in[F]_\varphi$, we have $\Gamma\vdash\pi:(F\Rightarrow G)_\varphi=F_\varphi\Rightarrow G_\varphi$ and $\Gamma'\vdash\pi':F_\varphi$. Therefore $\Gamma\Gamma'\vdash\pi\pi':G_\varphi$. We conclude by induction hypothesis.
– if $D=\forall x.F$,
  then $\Gamma\vdash\pi:(\forall x.F)_\varphi$ and $(\Gamma,\pi)\notin[\forall x.F]_\varphi$. Then there exists $t\in\hat{T}$ such that $(\Gamma,\pi t)\notin[F]_{\varphi+\langle x,t\rangle})$. As $(\Gamma,\pi)\in[\forall x.F]_\varphi$, we have $\Gamma\vdash\pi:(\forall x.F)_\varphi$, therefore $\Gamma\vdash\pi t:(t/x)F_\varphi=F_{\varphi+\langle x,t\rangle}$. We conclude by induction hypothesis.

Proposition 2 (Completeness). *If the theory $\mathcal{L}_\equiv$ is strongly normalizing, then $\lfloor.\rfloor_.=\langle A,\varphi\rangle\mapsto\lfloor A\rfloor_\varphi$ is a $\mathcal{C}_\equiv$-model of this theory.*

Proof. By remark 6 and lemmas 6 and 11.

3.5 The substitution property

We finally prove one more property concerning $\lfloor.\rfloor_.$, about well-typed substitution. A property we will need in section 4.

Lemma 12. *If $\mathcal{L}_\equiv$ is strongly normalising, then for all $E\in\mathcal{C}_\equiv$, $\alpha\in\mathcal{X}$, $\pi,\pi'\in\mathcal{T}$, $B\in\mathcal{P}$, $\varphi\in\mathrm{VAL}(B)$, and Γ,Γ' contexts such that $(\Gamma,\alpha)\in E$, $(\Gamma',\pi')\in E$ and $(\Gamma,\pi)\in[B]_\varphi$ (resp. $\lfloor B\rfloor_\varphi$) then $(\Gamma\Gamma',(\pi'/\alpha)\pi)\in[B]_\varphi$ (resp. $\lfloor B\rfloor_\varphi$).*

Proof. Notice that if the $[.]_.$ version of this lemma is true, then also is the $\lfloor . \rfloor_.$ version. Let us prove the $[.]_.$ version by induction on A.

- If A is atomic, we have $\Gamma \vdash \pi : B_\varphi$, $\Gamma \vdash \alpha : A_E$ and $\Gamma' \vdash \pi' : A_E$, therefore, $\Gamma\Gamma' \vdash (\pi'/\alpha)\pi : B_\varphi$. Moreover $\mathcal{L}_\equiv$ is strongly normalizing, then $(\pi'/\alpha)\pi \in SN$ and $(\Gamma\Gamma', (\pi'/\alpha)\pi) \in [B]_\varphi$.
- If $B = C \Rightarrow D$, then $(\Gamma, \pi) \in [C]_\varphi \mathring{\Rightarrow} [D]_\varphi$. Let $(\Gamma'', \tau) \in [C]_\varphi$ such that u doesn't contain α (by α-conversion). Then $(\Gamma\Gamma'', \pi\tau) \in [D]_\varphi$ therefore $(\Gamma\Gamma'\Gamma'', (\pi'/\alpha)(\pi\tau)) = (\Gamma\Gamma'\Gamma'', (\pi'/\alpha)\pi\ \tau) \in [D]_\varphi$, by induction hypothesis. Finally, $(\Gamma\Gamma', (\pi'/\alpha)\pi) \in [C]_\varphi \mathring{\Rightarrow} [D]_\varphi = [B]_\varphi$.
- If $B = \forall x.C$, then $(\Gamma, \pi) \in \mathring{\forall}_T(t \mapsto \lfloor C \rfloor_{\varphi+<x,t>})$. Let $t \in \hat{T}$, then $(\Gamma, \pi t) \in [C]_{\varphi+<x,t>}$. Therefore $(\Gamma\Gamma', (\pi'/\alpha)(\pi t)) = (\Gamma\Gamma', (\pi'/\alpha)\pi t) \in [C]_{\varphi+<x,t>}$, by induction hypothesis. Finally $(\Gamma\Gamma', (\pi'/\alpha)\pi) \in [B]_\varphi$.

Corollary 1. *If $\mathcal{L}_\equiv$ is strongly normalising,*
for all $E \in \mathcal{C}_\equiv$, $A \in \mathcal{P}$, $\varphi \in$ Val(A), $(\Gamma, \pi) \in \mathcal{U}$, $\alpha \in \mathcal{X}$,
if $(\Gamma, \alpha) \in E$ and $(\Gamma, \pi\alpha) \in \lfloor A \rfloor_\varphi$, then $(\Gamma, \pi) \in E \mathring{\Rightarrow} \lfloor A \rfloor_\varphi$.

We say that $\lfloor A \rfloor_\varphi$ satisfies the substitution property.

4 From $\mathcal{C}_\equiv$ to $\mathcal{C}'$

We have introduced, for each theory $\mathcal{L}_\equiv$ a LDTVA $\mathcal{C}_\equiv$ such that having a $\mathcal{C}_\equiv$-valued model is a complete semantic condition for strong normalization property. But each LDTVA $\mathcal{C}_\equiv$ depends on the congruence relation $\equiv$, while we want to define a sufficient and necessary semantics for *all* strongly normalizing theories expressed in minimal deduction modulo.

Let us now introduce $\mathcal{C}'$, an algebra which does not depend anymore on $\equiv$, and a morphism of algebras from each $\mathcal{C}_\equiv$ of strongly normalizing $\equiv$ to $\mathcal{C}'$, in order to prove that having a $\mathcal{C}'$-valued model is also a complete semantic condition for strong normalization property. Actually, we build a more general morphism: from each sub-LDTVA satisfying the substitution property of each $\mathcal{C}_\equiv$ to $\mathcal{C}'$.

We will consider for the domain of $\mathcal{C}'$ (wich we will also call $\mathcal{C}'$), the subsets of $\mathcal{T}$ which verify the usual properties (CR$_1$), (CR$_2$) of reducibility candidates, and a modified version of (CR$_3$), which gives a solution to avoid not well-typed proof-terms without talking about typing: as we said before, the problem of usual reducibility candidates comes from normal neutral not well-typed terms. In $\mathcal{C}_\equiv$, we avoid not well-typed terms while in $\mathcal{C}'$, the main idea is to avoid normal proof-terms (in our adaptation of (CR$_3$)).

4.1 $\mathcal{C}'$, yet another algebra of candidates.

Definition 19.
For all sets E of proof-terms, we define the following properties :

(CR$_1$) *For all $\pi \in E$, $\pi \in SN$.*

(CR_2) *For all $\pi \in E$, for all $\pi' \in \mathcal{T}$ such that $\pi \to \pi'$, then $\pi' \in E$.*

(CR'_3) *for all $n \in \mathbb{N}$, for all $\nu, \mu_1, \ldots, \mu_n \in \mathcal{T}$, if*

- *for all $i \leq n$, μ_i is neutral and not normal,*
- *$\forall \rho_1, \ldots, \rho_n \in \mathcal{T}$ such that for all $i \leq n$, $\mu_i \to \rho_i$, $(\rho_i/\alpha_i)_i \nu \in E$*

then $(\mu_i/\alpha_i)\nu \in E$.

Definition 20 **($\tilde{\Rightarrow}$).**
For all $E, F \subseteq \mathcal{T}$, $E \tilde{\Rightarrow} F = \{\pi \in SN$ such that for all $\pi' \in E$, $\pi\pi' \in F\}$.

Lemma 13. *$\tilde{\Rightarrow}$ is a function from $\mathcal{C}' \times \mathcal{C}'$ to $\mathcal{C}'$.*

Proof. Let $E, F \in \mathcal{C}'$ and $\pi \in E \tilde{\Rightarrow} F$,

(CR$_1$) $\pi \in SN$, by definition.
(CR$_2$) If ρ is a one-step reduct of π, then for all $\pi' \in E$, $\rho\pi'$ is a one-step reduct of $\pi\pi'$.
(CR$'_3$) If there exists $\nu, \mu_1, \ldots, \mu_n \in \mathcal{T}$, such that each μ_i is neutral not normal, $\tau = (\mu_i/\alpha_i)_i\ \nu$ and for all $(\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_i \to \rho_i$, then $(\rho_i/\alpha_i)_i\ \nu \in E \tilde{\Rightarrow} F$. Then, for all $\pi' \in E$, $\tau\pi' = (\mu_i/\alpha_i)_i\ \nu\pi' = (\mu_i/\alpha_i)_i\ \nu'$ with $\nu' = \nu\pi'$. And for all $(\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_i \to \rho_i$, we have $(\rho_i/\alpha_i)_i\ \nu' = (\rho_i/\alpha_i)_i\ \nu\ \pi' \in F$ by hypothesis, therefore $\tau\pi' \in F$ as it satifies (CR$'_3$). And finally, $\tau \in E \tilde{\Rightarrow} F$.

Definition 21 **($\tilde{\mathcal{A}}_T$).**
For all sorts T, $\tilde{\mathcal{A}}_T = \hat{T} \mapsto \mathcal{C}'$.

Definition 22 **($\tilde{\forall}_T$).** *For all sorts T and function $f \in \tilde{\mathcal{A}}_T$,*
$\tilde{\forall}_T.f = \{\pi \in \mathcal{T}$ such that for all $t \in \hat{T}$, $\pi t \in f(t)\}$

Lemma 14. *For all sorts T, $\tilde{\forall}_T$ is a function from $\tilde{\mathcal{A}}_T$ to $\mathcal{C}'$.*

Proof. Let T be a sort, $f \in \tilde{\mathcal{A}}_T$ and $\pi \in \tilde{\forall}_T.f$.

(CR$_1$) Let $t \in \hat{T}$ ($\neq \emptyset$), then $\pi t \in f(t) \in \mathcal{C}'$, therefore $\pi t \in SN$ and so does π.
(CR$_2$) Let π' such that $\pi \to \pi'$. Then for all $t \in \hat{T}$, $\pi' t$ is a one-step reduct of πt.
(CR$'_3$) If there exists $\nu, \mu_1, \ldots, \mu_n \in \mathcal{T}$, such that each μ_i is neutral not normal, $\tau = (\mu_i/\alpha_i)_i\ \nu$ and for all $(\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_i \to \rho_i$, then $(\rho_i/\alpha_i)_i\ \nu \in \tilde{\forall}_T.f$. Then, for all $t \in \hat{T}$, $\tau t = (\mu_i/\alpha_i)_i\ \nu t = (\mu_i/\alpha_i)_i\ \nu'$ with $\nu' = \nu t$. And for all $(\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_i \to \rho_i$, we have $(\rho_i/\alpha_i)_i\ \nu' = (\rho_i/\alpha_i)_i\ \nu\ t \in f(t)$ by hypothesis, therefore $\tau t \in f(t)$ as it satisfies (CR$'_3$). And finally, $\tau \in \tilde{\forall}_T.f$.

Definition 23 **($\mathcal{C}'$).** *$\mathcal{C}'$ is the* LDTVA *$\langle \mathcal{C}', \tilde{\Rightarrow}, (\tilde{\mathcal{A}}_T), (\tilde{\forall}_T) \rangle$.*

4.2 Building a function from $\mathcal{C}_{\equiv}$ to $\mathcal{C}'$

We build a function $Cl(.)$ from $\mathcal{C}_{\equiv}$ to $\mathcal{C}'$, mapping $\mathcal{C}_{\equiv}$-valued models satisfying the substitution property to $\mathcal{C}'$-valued models of the theory $\mathcal{L}_{\equiv}$. We first set a (big enough) context Δ, in order to make elements of the image of an element of $\mathcal{C}_{\equiv}$ under $Cl(.)$ be well-typed (by a same proposition), in the same context. Then, we extend these image sets to make them satisfy (CR$'_3$). Finally, given an image set, there exists a proposition such that for each reduction sequence from every element of this set, there exists a step of the reduction such that the reduct is typable by the proposition in Δ.

Definition 24 (Δ). *We consider a context which contains an infinite number of variables for each proposition.* $\Delta = (\beta_i^A : A)_{A\in\mathcal{P}, i\in\mathbb{N}}$.

Definition 25 (Leaves).
The leaves *of a proof-term π are its first reducts which are normal or not neutral. (ρ is a leaf of π if and only if ρ is normal or not neutral and there exists $n \geq 0$ and $\pi_1 \ldots \pi_{n-1}$ neutral not normal terms such that $\pi = \pi_1 \to \ldots \to \pi_{n-1} \to \rho$). We call $\mathcal{L}(\pi)$ the set of leaves of π.*

Remark 7. The only leaf of a normal or not neutral proof-term is itself.
If π is a neutral non-normal proof-term, then $\rho \in \mathcal{L}(\pi)$ if and only if there exists a one-step reduct π' of π such that $\rho \in \mathcal{L}(\pi')$.

Definition 26 (Closure). *For all $E \subseteq \mathcal{U}$, we define $Cl(E)$ as follows : for all $k \in \mathbb{N}$,*

– $Cl^0(E) = \{\pi \in \mathcal{T}$ *such that* $(\Delta, \pi) \in E\}$
– $Cl^{k+1}(E) = \{\pi \in \mathcal{T}$, *such that* $\exists n \in \mathbb{N}$:
$\exists \nu_\pi \in \mathcal{T}, \exists (\mu_i)_{i\leq n} \subseteq SN$, *each neutral not normal s.t.*
$\pi = (\mu_i/\alpha_i)_{i\leq n}\ \nu_\pi$ *and* $\forall (\rho_i)_{i\leq n} \subseteq \mathcal{T}$, *s.t.* $\forall i \leq n, \rho_i \in \mathcal{L}(\mu_i)$,
we have $(\rho_i/\alpha_i)_{i\leq n}\ \nu_\pi \in Cl^k(E)\}$
– $Cl(E) = \cup_{j\in\mathbb{N}} Cl^j(E)$

Remark 8. Notice that for all $E \in \mathcal{C}_{\equiv}$ and $k \in \mathbb{N}$, if $E \neq \emptyset$ then $Cl^0(E) \neq \emptyset$, therefore $Cl(E) \neq \emptyset$ and $Cl^k(E) \subseteq Cl^{k+1}(E)$.

Let us now prove that $Cl(.)$ maps each element of $\mathcal{C}_{\equiv}$ to an element of $\mathcal{C}'$.

Lemma 15.
For all $\pi \in SN$, if π is not isolated then there exists an abstraction τ such that for all abstractions τ' such that $\pi \to^* \tau'$, we have $\tau \to^* \tau'$.
We call τ the primary leaf *of π.*

Remark 9. This definition of *primary leaf* is very closed to C. Riba's one of *princpal reduct* [16]. The only difference is that in our case, the primary leaf has to be an abstraction.

Proof. If π is not isolated, then all reductions sequences from π reach a not neutral proof-term, by confluency. Notice that the not-neutral reducts of π are either all proof-abstractions, either all term-abstractions, by confluency. In particular, the head-reduction sequence from π reach a not-neutral proof-term. Let τ be the first non-neutral proof-term reached in this reductions sequence. If π is not neutral, then $\pi = \tau$, therefore all non-neutral reducts of π are obviously reducts of $\pi = \tau$. Otherwise, π is neutral and has the form $\rho\ \theta_1 \dots \theta_n$, with each θ_i a term or a proof-term, $n > 0$ (as π is neutral), and ρ is not neutral (as π is not isolated). We suppose, in the following that ρ is a proof-abstraction $\lambda\alpha.\rho'$, then θ_1 is a proof-term (the proof is the same in the case of a term-abstraction). Let us prove by induction on the maximal length of a reductions sequence from π ($\in SN$), that each not neutral reduct of π is a reduct of τ. If this maximal length is equal to zero, then π cannot be neutral. Otherwise, let $\pi', \pi'' \in \mathcal{T}$ such that π'' is not neutral and $\pi = (\lambda\alpha.\rho')\ \theta_1 \dots \theta_n \to \pi' \to^* \pi''$.

- If $\pi' = (\theta_1/\alpha)\rho'\ \theta_2 \dots \theta_n$, then π' is the head-one-step-reduct of π, therefore π'' is a reduct of τ, by induction hypothesis.
- If $\pi' = (\lambda\alpha.\rho'')\ \theta_1 \dots \theta_n$, with $\rho' \to \rho''$, let τ' be the first not neutral term reached in the head-reduction of $(\theta_1/\alpha)\rho''\ \theta_2 \dots \theta_n$ (then τ' is also the first not neutral term reached in the head-reduction of π'). By induction hypothesis, τ' is a reduct of τ. Moreover, π'' is a reduct of τ' by induction hypothesis (as τ' is the first not neutral head-reduct of $(\theta_1/\alpha)\rho'\ \theta_2 \dots \theta_n$). Finally, π'' is a reduct of τ.
- If $\pi' = (\lambda\alpha.\rho')\ \theta_1 \dots \theta_{i-1}\ \theta'_i\ \theta_{i+1} \dots \theta_n$, we use the same sort of argument than in the previous point.

Definition 27 **($\pi \shortparallel \alpha$).** *For all $\alpha \in \mathcal{X}$ and $\pi \in \mathcal{T}$, we write $\pi \shortparallel \alpha$ the number of occurrences of α in π.*

Definition 28 **($\mathcal{K}$).**
$\mathcal{K} = \{\ \langle \nu, n, (\mu_1, \dots, \mu_n)\rangle$ such that
. $n \in \mathbb{N}$, $\nu \in \mathcal{T}$, $\mu_1 \dots \mu_n \in SN$
. for all $i \leq n$, $\nu \shortparallel \alpha_i \leq 1$
. for all $(\rho_i)_i$ each respectively in $\mathcal{L}(\mu_i)$, $(\rho_i/\alpha_i)_i\nu \in SN\ \}$

Definition 29 **($\rightharpoondown$).**
Let $\delta = \langle \nu, n, (\mu_1, \dots, \mu_n)\rangle$ and $\delta' = \langle \nu', n', (\mu'_1, \dots, \mu'_n)\rangle$ in $\mathcal{K}$.
We say that $\delta \rightharpoondown \delta'$ if and only if:

(a) $\nu = \nu'$ and there exists $i_0 \leq n$ such that for all $i \neq i_0$, $\mu_i = \mu'_i$, μ_{i_0} is neutral and $\mu_{i_0} \to \mu'_{i_0}$ or
(b) there exists $i_0 \leq n$ such that μ_{i_0} is not neutral, $\nu' = (\mu_{i_0}/\alpha_{i_0})\nu$, $n' = n - 1$ and $\mu'_1 \dots \mu'_{n'} = \mu_1 \dots \mu_{i_0-1}\ \mu_{i_0+1} \dots \mu_n$ or
(c) $\nu \to \nu'$ and the μ'_i are copies of the μ_i resulting of the linearization of the occurrences of the variables α_i in ν'.

Definition 30 **($\ell(\pi)$).** *For all $\pi \in SN$, we define $\ell(\pi)$ as follows: if π is isolated, $\ell(\pi) = \alpha_0$ (a special variable), otherwise, $\ell(\pi)$ is the primary leaf of π.*

Definition 31 **($||\delta||$).** *For all $\delta = \langle \nu, n, (\mu_1, \ldots, \mu_n) \rangle \in \mathcal{K}$, $||\delta|| = (\ell(\mu_i)/\alpha_i)_i\nu$.*

Remark 10. For all $\delta \in \mathcal{K}$, $||\delta|| \in SN$, by definition.

Lemma 16. *For all $\delta, \delta' \in \mathcal{K}$, if $\delta \rightharpoondown \delta'$ then $||\delta|| \to^* ||\delta'||$.*

Proof. By case on $\delta \rightharpoondown \delta'$.

(a) If the μ_{i_0} which is reduced (on μ'_{i_0}) is isolated then $\ell(\mu_{i_0}) = \ell(\mu'_{i_0}) = \alpha_0$, therefore $||\delta|| = ||\delta'||$. Otherwise $\ell(\mu_{i_0}) \to^* \ell(\mu'_{i_0})$, by lemma 15, therefore $||\delta|| \to^* ||\delta'||$.
(b) In this case, $||\delta|| = ||\delta'||$.
(c) In this case, $||\delta|| \to ||\delta'||$ (as $\nu \to \nu'$).

Lemma 17. *All $\rightharpoondown$-reductions sequences from an element δ of $\mathcal{K}$ are finite.*

Proof. As all the μ_i are in SN (and n is finite), there can only be a finite number of consecutive (a) and (b) reductions. As $||\delta|| \in SN$, there can only be a finite number of (c) reductions from δ, by lemma 16. Hence there cannot be an infinite $\rightharpoondown$-reductions sequence from δ.

Then we can use the previous lemmas in order to prove that $Cl(.)$ maps elements of $\mathcal{C}_\equiv$ to elements of $\mathcal{C}'$.

Proposition 3.
For all $E \in \mathcal{C}_\equiv$, $Cl(E) \in \mathcal{C}'$.

Proof. Let $E \in \mathcal{C}_\equiv$.

(CR$_2$) Let $\pi \in Cl(E)$ and $\pi' \in \mathcal{T}$ such that $\pi \to \pi'$. Then there exists (a minimal) $k \in \mathbb{N}$ such that $\pi \in Cl^k(E)$. By induction on k.
- If $k = 0$, then $(\Delta, \pi) \in E$, therefore $(\Delta, \pi') \in E$ as it satisfies (CR$_{2_\equiv}$).
- If $k > 0$, then $\pi = (\mu_i/\alpha_i)_i\nu$ with each μ_i in SN, neutral and not normal, and such that $\forall i \leq n$, and for all $(\rho_i)_i$ such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, we have $(\rho_i/\alpha_i)_i\nu \in Cl^{k-1}(E)$. We suppose that for all $i \leq n$, $\nu \shortparallel \alpha_i \leq 1$. As each μ_i is neutral:
  . Either $\pi' = (\mu'_{i_0}/\alpha_{i_0})(\mu_i/\alpha_i)_{i \neq i_0}\nu$, with $\mu_{i_0} \to \mu'_{i_0}$. In this case,
    . if $\mu'_{i_0} \in \mathcal{L}(\mu_{i_0})$ then $\pi' = (\mu_i/\alpha_I)_{i \neq i_0}\nu''$, with $\nu'' = (\mu'_{i_0}/\alpha_{i_0})\nu$, , and for all $(\rho_i)_i$ such that $\forall i \neq i_0$, $\rho_i \in \mathcal{L}(\mu_i)$, we have $(\rho_i/\alpha_i)_i\nu'' \in Cl^{k-1}(E)$, hence $\pi' \in Cl^k(E)$.
    . Otherwise, μ'_{i_0} is neutral, not normal and all its leaves are leaves of μ_{i_0}, hence $\pi' \in Cl^k(E)$.
  . Either $\pi' = (\mu_i/\alpha_i)_i\nu$ with $\nu \to \nu'$. In this case, we conclude by the fact that $Cl^{k-1}(E)$ satisfies (CR$_2$) by induction hypothesis.

(CR$_1$) Let $\pi \in Cl(E)$, then there exists (a minimal) $k \in \mathbb{N}$ such that $\pi \in Cl^k(E)$. By induction on k.
- If $k = 0$, then $(\Delta, \pi) \in E$, therefore $\pi \in SN$ as E satisfies (CR$_{1_\equiv}$).

- If $k > 0$, then $\pi = (\mu_i/\alpha_i)_i\nu$ with each μ_i in SN, neutral and not normal, and such that $\forall i \leq n$, and $\forall(\rho_i)_i$ such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, we have $(\rho_i/\alpha_i)_i\nu \in Cl^{k-1}(E) \subseteq SN$, by induction hypothesis. Then, if we suppose that for all $i \leq n$, $\nu||\alpha_i \leq 1$, if $\pi \to \pi'$, and if we write $\delta_\pi = \langle \nu, n, (\mu_1, \ldots, \mu_n)\rangle$ (and the same for $\delta_{\pi'}$, as $\pi' \in Cl^k(E)$ as explained in the previous point), then $\delta_\pi \rightharpoondown \delta_{\pi'}$. Hence all reductions sequences from π are finite, by lemma 17.

(CR_3') Let $n \in \mathbb{N}$, $\nu, \mu_1, \ldots, \mu_n \in \mathcal{T}$, such that for all $i \leq n$, μ_i is neutral and not normal, and for all $\mu_1', \ldots, \mu_n' \in \mathcal{T}$ such that $\forall i \leq n$, $\mu_i \to \mu_i'$, $(\mu_i'/\alpha_i)_i\nu \in Cl(E)$. As the number of one-step reducts of a term is finite, there exists $k \in \mathbb{N}$, such that for all $\mu_1', \ldots, \mu_n' \in \mathcal{T}$ such that $\forall i \leq n$, $\mu_i \to \mu_i'$, we have $(\mu_i'/\alpha_i)_i\nu \in Cl^k(E)$. Therefore, for all $\rho_1 \ldots \rho_n$ each respectively a leaf of $\mu_1 \ldots \mu_n$, $(\rho_i/\alpha_i)_{i\leq n}\nu \in Cl^k(E)$ as it satisfies (CR_2) and each μ_i is neutral not normal. Finally, $(\mu_i/\alpha_i)_i\nu \in Cl^{k+1}(E)$.

4.3 Proving that the function Cl is a morphism

⇒-morphism

We prove now that for all $E, F \in \mathcal{C}_{\equiv}$ such that F satisfies the substitution property, we have $Cl(E \mathring{\Rightarrow} F) = Cl(E) \mathring{\Rightarrow} Cl(F)$.

Lemma 18. *For all $E \subseteq \mathcal{T}$ and $\pi \in \mathcal{T}$,*
If $\pi \in SN$, π is neutral not normal and $\forall \rho \in \mathcal{L}(\pi)$, $\rho \in Cl(E)$, then $\pi \in Cl(E)$

Proof. As $\pi \in SN$, $\mathcal{L}(\pi)$ is defined and finite.
And, if we call $k_m = \max\{\min\{k, \rho \in Cl^k(E)\},\ \rho \in \mathcal{L}(\pi)\}$,
then $\pi \in Cl^{k_m+1}(E) \subseteq Cl(E)$.

Remark 11. In the same way, if there exists $\nu_\pi \in \mathcal{T}$, and $(\mu_i)_i \subseteq SN$, each neutral not normal such that $\pi = (\mu_i/\alpha_i)_i\ \nu_\pi$ and $\forall (\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n, \rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_i\ \nu_\pi \in Cl(E)$, we have $\pi \in Cl(E)$.

Lemma 19. *For all $E, F \in \mathcal{C}_{\equiv}$,*
if F satisfies the substitution property, $(\Delta, \alpha) \in E$, and $(\alpha/\beta)\pi \in Cl(F)$
then $\lambda\beta.\pi \in Cl(E \mathring{\Rightarrow} F)$.

Proof. There exists a minimal k such that $(\alpha/\beta)\pi \in Cl^k(F)$. By induction on k.

- if $k = 0$ then $(\Delta, \pi) \in F$ and π is normal. As $(\Delta, \alpha) \in E$, we have $(\Delta, (\lambda\beta.\pi)\alpha) \in F$, by ($\mathrm{CR}_{3_\equiv}$). Therefore $(\Delta, \lambda\beta.\pi) \in E \mathring{\Rightarrow} F$, by substitution property. And $\lambda\beta.\pi \in Cl^0(E \mathring{\Rightarrow} F)$, as it is normal.
- if $k > 0$, then $(\alpha/\beta)\pi = (\mu_i/\alpha_i)_{i\leq n}\nu$ with each μ_i in SN, neutral and not normal, and such that $\forall i \leq n$, and $\forall(\rho_i)_{i\leq n}$ such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, we have $(\rho_i/\alpha_i)_{i\leq n}\nu \in Cl^{k-1}(F)$, therefore $\lambda\beta.(\rho_i/\alpha_i)_{i\leq n}\nu = (\rho_i/\alpha_i)_{i\leq n}\lambda\alpha.\nu \in Cl(E \mathring{\Rightarrow} F)$, by induction hypothesis. Therefore $\lambda\beta.\pi \in Cl(E \mathring{\Rightarrow} F)$ by remark 11.

Lemma 20. *For all $E, F \in \mathcal{C}_{\equiv}$ and $\pi \in Cl(E) \mathrel{\tilde{\Rightarrow}} Cl(F)$,*
π cannot reduce to a term-abstraction.

Proof. Let $\pi \in Cl(E) \mathrel{\tilde{\Rightarrow}} Cl(F)$, then $\pi \in SN$. If π reduces to a term-abstraction, then its normal form is a term-abstraction $\lambda x.\rho$, by confluency. Let $\alpha \in \mathcal{X}$ such that $(\Delta, \alpha) \in E$, then $\alpha \in Cl(E)$ and $\pi\alpha \in Cl(F)$, therefore $(\lambda x.\rho)\alpha \in Cl(F)$, as F satisfies (CR$_2$). Moreover, $(\lambda x.\rho)\alpha$ is normal, therefore $(\lambda x.\rho)\alpha \in Cl^0(F)$. Hence $(\Delta, (\lambda x.\rho)\alpha) \in F$ and $\Delta \vdash (\lambda x.\rho)\alpha : A_F$. That's absurd.

Proposition 4. *For all $E, F \in \mathcal{C}_{\equiv}$,*
if F satisfies the subsitution property, then $Cl(E \mathrel{\mathring{\Rightarrow}} F) = Cl(E) \mathrel{\tilde{\Rightarrow}} Cl(F)$.

Proof. $\subseteq$ Let $\pi \in Cl(E \mathrel{\mathring{\Rightarrow}} F)$,

then $\pi \in SN$ by (CR$_1$). Moreover there exists (a minimal) $k \in \mathbb{N}$, such that $\pi \in Cl^k(E \mathrel{\mathring{\Rightarrow}} F)$. Let $\pi' \in Cl(E)$, then there exists (a minimal) $j \in \mathbb{N}$, such that $\pi' \in Cl^j(E)$. Let us show that $\pi\pi' \in Cl(F)$ by induction on $k + j$.

- If $k + j = 0$ then $(\Delta, \pi) \in E \mathrel{\mathring{\Rightarrow}} F$ and $(\Delta, \pi') \in E$ therefore $(\Delta, \pi\pi') \in F$ and $\pi\pi' \in Cl^0(F)$.
- If $k > 0$, then there exists $\nu_\pi \in \mathcal{T}$, and $(\mu_i)_i \subseteq SN$, each neutral not normal such that $\pi = (\mu_i/\alpha_i)_i\ \nu_\pi$ and $\forall (\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n, \rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_i\ \nu_\pi \in Cl^{k-1}(E \mathrel{\mathring{\Rightarrow}} F)$.
  Therefore $(\rho_i/\alpha_i)_i\ (\nu_\pi\ \pi') = (\rho_i/\alpha_i)_i\ \nu_\pi\ \pi' \in Cl(F)$ by induction hypothesis. Hence $\pi\pi' \in Cl(F)$, as it satisfies (CR$'_3$).
- If $j > 0$, then there exists $\nu_{\pi'} \in \mathcal{T}$, and $(\mu_i)_i \subseteq SN$, each neutral not normal such that $\pi' = (\mu_i/\alpha_i)_i\ \nu_{\pi'}$ and $\forall (\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_i\ \nu_{\pi'} \in Cl^{j-1}(E)$.
  Therefore $(\rho_i/\alpha_i)_i\ (\pi\ \nu_{\pi'}) = (\rho_i/\alpha_i)_i\ \pi\ \nu_{\pi'} \in Cl(F)$ by induction hypothesis. Hence $\pi\pi' \in Cl(F)$, as it satisfies (CR$'_3$).

$\supseteq$ Let $\pi \in Cl(E) \mathrel{\tilde{\Rightarrow}} Cl(F)$. then $\pi \in SN$ and for all $\pi' \in Cl(E)$, $\pi\pi' \in Cl(F)$. By lemma 20, π cannot reduce to a term-abstraction.

- If π is a proof-abstraction $\lambda\alpha.\pi'$, let $\beta \in \mathcal{X}$ such that $\Delta \vdash \beta : A_E$, then $(\lambda\alpha.\pi')\beta \in Cl(F)$ and so does $(\beta/\alpha)\pi'$, by (CR$_2$).Therefore $\pi \in Cl(E \mathrel{\mathring{\Rightarrow}} F)$ by lemma 19.
- If π is neutral and normal, let $\alpha \in \mathcal{X}$ such that $\Delta \vdash \alpha : A_E$, then $\pi\alpha \in Cl(F)$. Moreover π is neutral and normal, therefore $\pi\alpha$ is normal, hence $\pi\alpha \in Cl^0(F)$, *i.e.* $(\Delta, \pi\alpha) \in F$, with $(\Delta, \alpha) \in E$, therefore $(\Delta, \pi) \in E \mathrel{\mathring{\Rightarrow}} F$, as F satisfies the substitution property.
  Finally, $\pi \in Cl^0(E \mathrel{\mathring{\Rightarrow}} F)$, as it is normal.
- Otherwise, $\pi \in SN$, is neutral and not normal. All its leaves are either neutral, either proof-abstractions, by lemma 20. And all these leaves are in $Cl(E) \mathrel{\tilde{\Rightarrow}} Cl(F)$, as it satisfies (CR$_2$), therefore they also are in $Cl(E \mathrel{\mathring{\Rightarrow}} F)$, as we saw in the previous points. Finally, $\pi \in Cl(E \mathrel{\mathring{\Rightarrow}} F)$, by lemma 18.

$\forall$-morphism

We prove now that for all sorts T and $f \in \mathring{\mathcal{A}}_T$, $Cl(\mathring{\forall}_T f) = \tilde{\forall}_T\ Cl \circ f$. Notice that for all functions $f \in \mathring{\mathcal{A}}_T$, $Cl \circ f \in \tilde{\mathcal{A}}_T$.

Lemma 21. *For all $E \in \mathcal{C}_{\equiv}$, $k \in \mathbb{N}$, terms t, term-variables x, proof-terms π', if $(t/x)\pi \in Cl^k(E)$, then $(\lambda x.\pi)t \in Cl^k(E)$.*

Proof. By induction on k.
- If $k = 0$, by $(\mathrm{CR}_{3_{\equiv}})$.
- If $k > 0$, by induction hypothesis.

Lemma 22. *For all $\pi \in \mathcal{T}$ and $f \in \mathring{\mathcal{A}}_T$, if $\pi \in \tilde{\forall}_T\ Cl \circ f$ then there exists $k \in \mathbb{N}$ such that $\pi \in \tilde{\forall}_T\ Cl^k \circ f$.*

Proof. For all $E \in \mathcal{C}_{\equiv}$, if $\pi \in Cl(E)$ then $\pi \in SN$ and if k is the maximal length of a reductions sequence from π then $\pi \in Cl^k(E)$. Then, as the maximal length of reductions sequence from πt is the same for all $t \in \hat{T}$, if we note l this maximal length, we have, for all $t \in \hat{T}$, $\pi t \in Cl^l \circ f(t)$, therefore $\pi \in \tilde{\forall}_T Cl^l \circ f$.

Proposition 5. *For all sorts T and $f \in \mathring{\mathcal{A}}_T$, $Cl(\mathring{\forall}_T f) = \tilde{\forall}_T\ Cl \circ f$.*

Proof. $\subseteq$ Let $\pi \in Cl(\mathring{\forall}_T f)$, then there exists (a minimal) $k \in \mathbb{N}$ such that $\pi \in Cl^k(\mathring{\forall}_T f)$. By induction on k.
  - If $k = 0$, $(\Delta, \pi) \in \mathring{\forall}_T f$ and $\pi \in SN$, then for all $t \in \hat{T}$, $(\Delta, \pi t) \in f(t)$, hence $\pi t \in Cl^0 \circ f(t)$. And $\pi \in \tilde{\forall}_T\ Cl \circ f$.
  - If $k > 0$, then $\pi = (\mu_i/\alpha_i)_i \nu$, with each μ_i neutral not normal and such that for all $(\rho_i)_{i \leq n}$ each respectively a leaf of μ_i, we have $(\rho_i/\alpha_i)_i \nu \in Cl^{k-1}(\mathring{\forall}_T f) \subseteq \tilde{\forall}_T\ Cl \circ f$, by induction hypothesis. Let $t \in \hat{T}$, then if we write $\nu' = \nu t$, we have $\pi t = (\mu_i/\alpha_i)_i \nu'$ and for all $(\rho_i)_{i \leq n}$ each respectively a leaf of μ_i, $(\rho_i/\alpha_i)_i \nu' = (\rho_i/\alpha_i)_i \nu\ t \in Cl \circ f(t)$. Therefore $\pi t \in Cl \circ f(t)$ by remark 11. Finally, $\pi \in \tilde{\forall}_T\ Cl \circ f$.

$\supseteq$ Let $\pi \in \tilde{\forall}_T\ Cl \circ f$, then, by lemma 22, there exists $k \in \mathbb{N}$ such that $\pi \in \tilde{\forall}_T Cl^k \circ f$. By induction on k.
  - If $k = 0$, then there exists $t \in \hat{T}$ such that $\pi t \in Cl^0 \circ f(t)$. Hence $(\Delta, \pi t) \in f(t)$ and πt is normal. Hence π is normal and for all $t' \in \hat{T}$, $\pi t'$ is also normal, therefore, as $\pi t' \in Cl \circ f(t)$, we have, in particular, $\pi t' \in Cl^0 \circ f(t)$. Finally, for all $t' \in \hat{T}$, $(\Delta, \pi t') \in f(t)$, therefore $(\Delta, \pi) \in \mathring{\forall}_T f$, and $\pi \in Cl^0(\mathring{\forall}_T f)$, as it is normal.
  - If $k > 0$, let $t \in \hat{T}$ such that $\pi t \in Cl^k \circ f(t)$. Therefore $\pi t = (\mu_i/\alpha_i)_i \nu$, with each μ_i neutral not normal and such that for all $(\rho_i)_{i \leq n}$ each respectively a leaf of μ_i, we have $(\rho_i/\alpha_i)_i \nu \in Cl^{k-1} \circ f(t)$.
    * If $\nu \neq \alpha_1$, then $\nu = \nu' t$, with $\pi = (\mu_i/\alpha_i)_i \nu'$, and for all $(\rho_i)_{i \leq n}$ each respectively a leaf of μ_i, we have $(\rho_i/\alpha_i)_i \nu' \in Cl(\mathring{\forall}_T f)$, by induction hypothesis. We conclude by lemma 18.
    * Otherwise, every leaf of πt is in $Cl^{k-1} \circ f(t)$. If π is isolated, then all its leaves ρ are neutral and normal, hence ρt is a leaf of πt, therefore $\rho \in Cl(\mathring{\forall}_T f)$, by induction hypothesis, and we conclude by lemma 18. If π reduces to $\lambda x.\pi'$ then all leaves of $(t/x)\pi'$ are in $Cl^{k-1} \circ f(t)$, therefore, for all leaves ρ of π', we have $(\lambda x.\rho)t \in Cl^{k-1} \circ f(t)$, by lemma 21, hence $\lambda x.\rho \in Cl(\mathring{\forall}_T f)$, by induction hypothesis. And finally, $\lambda x.\pi' \in Cl(\mathring{\forall}_T f)$, and so does π.

We finally get the following (second) completeness result:

Theorem 1.
If $\mathcal{L}_\equiv$ is strongly normalizing, then $Cl \circ \lfloor.\rfloor_.$ is a (non-empty) $\mathcal{C}'$-valued model of $\mathcal{L}_\equiv$.

Proof. By lemma 1 and propositions 1, 2, 3, 4, 5.

5 Soundness

We finally prove in this section, that having a $\mathcal{C}'$-valued model is also a sound condition for strongly normalizing theories $\mathcal{L}_\equiv$.

Lemma 23. *If $[\![.]\!]_.$ is a $\mathcal{C}'$-valued model of a theory $\mathcal{L}_\equiv$, then for all $A \in \mathcal{P}$, contexts Γ, $\varphi \in \text{VAL}(A) \cap \text{VAL}(\Gamma)$, $\pi \in \mathcal{T}$ and σ substitutions such that for all declarations $\alpha : B$ in Γ, $\sigma\alpha \in \mathcal{F}(B, \varphi)$, we have:*

$$\text{if } \Gamma \vdash \pi : A \text{ then } \sigma\varphi\pi \in [\![A]\!]_\varphi.$$

Proof. By induction on the length of the derivation of $\Gamma \vdash \pi : A$. By case on the last rule used. If the last rule used is :

- axiom: in this case, π is a variable α, and Γ contains a declaration $\alpha : B$ with $A \equiv B$ (therefore $A_\varphi \equiv |B|_\varphi$). Then $\sigma\varphi\pi = \sigma\alpha \in [\![B]\!]_\varphi = [\![A]\!]_\varphi$.
- $\Rightarrow$-intro: in this case, π is an abstraction $\lambda\alpha.\tau$, and we have $\Gamma, \alpha : B \vdash \tau : C$ with $A \equiv B \Rightarrow C$. Let σ' such that for all variables β declared in Γ, $\sigma'\beta = \sigma\beta$ and $\sigma'\alpha$ is an element of $[\![B]\!]_\varphi$. Then $\sigma'\varphi\tau \in [\![C]\!]_\varphi$ by induction hypothesis (and $\sigma'\varphi\tau$ is in SN, therefore $\sigma\varphi\pi$ is also in SN). Let $\pi' \in [\![B]\!]_\varphi$, we prove by induction on the sum of both maximal lengths of a reductions sequence from $\sigma\varphi(\lambda\alpha.\tau)$ and π' (each in SN) that every one-step reduct of the neutral not normal proof-term $\sigma\varphi(\lambda\alpha.\tau)\ \pi'$ is in $[\![C]\!]_\varphi$. If the one-step reduct is $\sigma\varphi(\pi'/\alpha)\tau$, we conclude by induction hypothesis (on the length of the derivation) as $\pi' \in [\![B]\!]_\varphi$. Otherwise, the reduction takes place either in $\sigma\varphi(\lambda\alpha.\tau)$, either in π'. We conclude by induction hypothesis on the sum of the maximal lengths of reductions sequence from $\sigma\varphi(\lambda\alpha.\tau)$ and π'. And the fact that both $[\![B]\!]_\varphi$ and $[\![B \Rightarrow C]\!]_\varphi$ satisfy (CR$_2$). Finally, $\sigma\varphi(\lambda\alpha.\tau)\ \pi' \in [\![C]\!]_\varphi$, as it satisfies (CR$_3$') and $\sigma\varphi(\lambda\alpha.\tau)\ \pi'$ is neutral, not normal. Hence $\sigma\varphi(\lambda\alpha.\tau) \in [\![B]\!]_\varphi \tilde{\Rightarrow} [\![C]\!]_\varphi = [\![B \Rightarrow C]\!]_\varphi = [\![A]\!]_\varphi$
- $\Rightarrow$-elim: in this case, π is an application $\rho\tau$, and we have $\Gamma \vdash \rho : C \equiv B \Rightarrow A$ and $\Gamma \vdash \tau : B$. Therefore, by induction hypothesis, $\sigma\varphi\rho \in [\![B \Rightarrow A]\!]_\varphi = [\![B]\!]_\varphi \tilde{\Rightarrow} [\![A]\!]_\varphi$ and $\sigma\varphi\tau \in [\![B]\!]_\varphi$. Therefore $\sigma\varphi(\rho\tau) \in [\![A]\!]_\varphi$.
- $\forall$-intro: in this case, π is a term abstraction $\lambda x.\pi'$ and we have $\Gamma \vdash \pi' : B$ with $A \equiv \forall x.B$. Let $t \in \hat{T}$ (with T the sort of x), and $\varphi' = \varphi + \langle x, t\rangle$. Then $\sigma\varphi'\pi' = \sigma\varphi(t/x)\pi' \in [\![B]\!]_{\varphi'}$, by induction hypothesis. Therefore, $\sigma\varphi(\lambda x.\pi') \in \tilde{\forall}_T(t \mapsto [\![B]\!]_{\varphi+\langle x,t\rangle} = [\![A]\!]_\varphi$ (by induction on the maximal length of a reductions sequence from πt, with $t \in \hat{T}$, using the fact that for all $t \in \hat{T}$, $[\![B]\!]_{\varphi+\langle x,t\rangle}$ satisfies (CR$_2$) and (CR$_3$').

– $\forall$-elim: in this case, π is an application ρt, and we have $\Gamma \vdash \rho : \forall x.B$ with $A = (t/x)B$ and $x \notin FV(\Gamma)$. By induction hypothesis, we have $\sigma\varphi\rho \in [\![\forall x.B, \varphi]\!] = \tilde{\forall}_T(t \mapsto [\![B]\!]\varphi + \langle x, t\rangle)$. Therefore $\sigma\varphi(\rho t) = \sigma\varphi\rho\ (\varphi t) \in [\![B]\!]_{\varphi+\langle x,\varphi t\rangle} = [\![(t/x)B]\!]_\varphi = [\![A]\!]_\varphi$

Theorem 2. *If $\mathcal{L}_\equiv$ has a $\mathcal{C}'$-valued model, then $\mathcal{L}_\equiv$ is strongly normalizing.*

Proof. If $\mathcal{F}$ is a $\mathcal{C}'$-valued model of $\equiv$ then for all typing judgement $\Gamma \vdash \pi : A$ and σ and φ as in the previous proposition, we have $\sigma\varphi\pi \in \mathcal{F}(A, \varphi)$ hence $\sigma\varphi\pi \in SN$, therefore $\pi \in SN$.

Conclusion

We have defined a refinement of truth values algebras which allows to build more precise models. Then we exhibited one of these truth values algebras $\mathcal{C}'$ such that having a $\mathcal{C}'$-valued model is a sound and complete condition for strongly normalizing theories.While soundness is an usual property, this completeness result is, up to our knowledge, the first for strongly normalizing theories, in deduction modulo.

We proved this completeness theorem in an original way : build a structure adapted to the congruence relation and then show that it is also adapted to connectives when the theory is strongly normalizing. This way, we are able to build an interpretation of propositions adapted to the congruence relation, even if the theory is not strongly normalizing.

In future work, we wish to extend this result to other logical frameworks with or without rewriting. For logical frameworks with rewriting, we want to extend first this result to (complete) Deduction modulo, and to $\lambda\Pi$-calculus modulo [2]. We also want to study how these language-dependent truth values algebras can help us in building models of logical frameworks with dependent types, as $\lambda\Pi$-calculus modulo, or Pure Type Systems.

References

1. A. Church, A formulation of the simple theory of types, *The Journal of Symbolic Logic*, 5:56–68, 1940.
2. D. Cousineau and G. Dowek, Embedding Pure Types Systems in the lambda Pi-calculus modulo, *Typed Lambda calculi and Applications*. Lecture Notes in Computer Science 4583, Springer. pp. 102-117. 2007.
3. G.Dowek. Truth values algebras and proof normalization. *Types for proofs and programs*. Lecture Notes in Computer Science 4502. 2007, pp. 110-124.
4. G.Dowek, T.Hardin, and C.Kirchner. Theorem proving modulo. *Journal of Automated Reasoning*, 31:32–72, 2003.
5. G.Dowek, T.Hardin, and C.Kirchner. HOL-lambda-sigma: an intentional first-order expression of higher-order logic. *Mathematical Structures in Computer Science*, 11:1–25, 2001.
6. G.Dowek and A. Miquel, Cut elimination for Zermelo set theory, *manuscript*, 2007.
7. G.Dowek and B.Werner. Proof normalization modulo. *The Journal of Symbolic Logic*, 68(4):1289–1316, 2003.
8. G.Dowek and B.Werner. Arithmetic as a theory modulo. J. Giesel (Ed.), *Term rewriting and applications*, Lecture Notes in Computer Science 3467, Springer-Verlag, 2005, pp. 423-437.
9. M. Fiore, G. Plotkin and D. Turi. Abstract syntax and variable binding. *14th Annual Symposium on Logic in Computer Science*, pages 193–202, 1999.
10. J.-Y. Girard. Une extension de l'interprétation de Gödel à l'analyse, et son application à l'élimination des coupures dans l'analyse et la théorie des types. In J.Fenstad, editor, 2^{nd} *Scandinavian Logic Symposium*, pages 63–92. North Holland, 1971.
11. K.Gödel. Über die Vollständigkeit des Logikkalküls. *Doctoral dissertation*, University Of Vienna. 1929.
12. M. Hamana, Universal Algebra for Termination of Higher-Order Rewriting, *16th International Conference on Rewriting Techniques and Applications*, Lecture Notes in Computer Science 3467, Springer, pp. 135-149, 2005.
13. O.Hermant. A model based cut elimination proof. In 2^{nd} *St-Petersbourg Days in Logic and Computability*, 2003.
14. O.Hermant. *Méthodes sémantiques en déduction modulo*. Doctoral Thesis. Université de Paris 7, 2005.
15. P.A.Melliès, B.Werner. A Generic Normalization Proof for Pure Type Systems. *Types for proofs and programs*. Lecture Notes in Computer Science 1512. 1996.
16. C. Riba. On the Stability by Union of Reducibility Candidates. *10th International Conference on Foundations of Software Science and Computational Structures*, pp 317-331. (2007)
17. W.W. Tait. Intentional interpretations of functionals of finite type I. *The Journal of Symbolic Logic*, 32:198–212, 1967.